OPENING OF A CUT IN AN ELASTIC PLANE UNDER THE ACTION OF A MOVING LOAD

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The problem of the steady motion of a slit of finite length along a cut in an elastic plane (the range of velocities is subsonic) is discussed. It reduces to the generalized Hilbert problem, whose solution has a different set of characteristics in the sub- and super-Rayleigh regimes of motion. The size of the separation region and the position of the slit relative to the load are unknown in advance and are determined in the course of the solution from additional physical conditions. First of all this produces the nonlinear nature of the problem, and secondly it significantly distinguishes this discussion from the Ioffe problem [1] from crack theory, which it resembles in its formulation. The solution for a crack of finite length is given for comparison.

1. A uniform compressive stress $-\sigma_\infty$ ($\sigma_\infty > 0$) and a dynamic load $-\sigma(x_1, t)$, which is applied symmetrically along the normal to the edges of the cut and is stationary in the moving coordinate system $x = x_1 - ct$, $y = x_2$ (t is the time), and which causes opening of the cut on the section $L = \{|x| < d\}$ (Fig. 1), act in an elastic plane with a cut along the x_1 axis. Attention is devoted below to a justification of the selected scheme of steady motion of a simply connected and thin slit in a cut material (within the framework of linear elasticity theory) and the constraints on the class of permissible loads associated with this problem. We shall postulate for the present that $\sigma(x)$ is a nonnegative finite function of the Hölder class [2] with the carrier $L_0 = \{\delta_1 < x < \delta_2\} \subset L$.

By virtue of the symmetry principle we shall discuss the problem only in the upper half plane. We shall seek a steady field of perturbed velocities $u_i(x, y)$ and a field of perturbed stresses $\sigma_{ij}(x, y)$ ($i, j = 1, 2$) in the region $D = \{|x| < \infty, y > 0\}$ and also the numerical parameters d and δ_1 not known in advance, which specify the length of the opening and the location of the load relative to the selected coordinate system under the following conditions on the boundary $y = 0$:

$$\begin{gathered}\sigma_{12} = 0\ (|x| < \infty),\ u_2 = 0\\ (x \in L_1 = \{|x| > d\}),\\ \sigma_{22} = \sigma_\infty - \sigma(x)\ (x \in L_0),\\ \sigma_{22} = \sigma_\infty\ (x \in L - L_0);\end{gathered} \tag{1.1}$$

$$\begin{gathered}\sigma_{22}^0 = \sigma_{22} - \sigma_\infty \leqslant 0\ (x \in L_1),\\ f(x) \geqslant 0\ (x \in L),\end{gathered} \tag{1.2}$$

$$f(x) = -\frac{1}{c}\int\limits_x^d u_2(x', 0)\,dx'\ (x \in L),\quad f(d) = f(-d) = 0.$$

At infinity we require the regular disappearance of the stresses

$$\sigma_{ij} < \mathrm{const}/|\mathbf{x}| \qquad (|\mathbf{x}| \to \infty,\ i, j = 1, 2), \tag{1.3}$$

and at the singular points behavior of the functions such that

$$0 \leqslant w_k < \infty\ (k = 1, 2), \tag{1.4}$$

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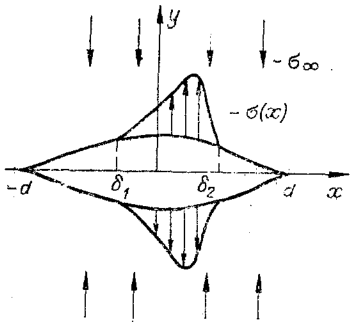

Fig. 1

where the power liberated at the points $x_1 = (-d, 0)$ and $x_2 = (d, 0)$, respectively, is denoted by w_k.

Conditions (1.2)-(1.4) are supplementary physical requirements to (1.1) which serve to isolate a unique solution. Inequalities (1.2) express, respectively, the condition of joining of the cut edges outside the slit (σ_{22}^0 is the total stress) and the nonintersection of the cut edges in the opening zone ($y = f(x)$ is the vertical shift of the upper edge). The energy requirements (1.4) mathematically limit the possible singularities of the solution at the replacement points of the boundary conditions and in addition permit the mechanical energy to disappear but not to emerge at these points. The latter is equivalent to the absence of concentrated forces at the singular points which produce positive work. The requirement of the energy flow be nonnegative plays an important role in the proof of the uniqueness theorem [3], and the example of [4] shows how nonuniqueness can arise otherwise.

In the time-independent problem of dynamic elasticity theory (two-dimensional strain) one can introduce the following representations of the desired functions in terms of two functions of a complex variable which coincide with the representations of [5] to within the accuracy of a factor:

$$
\begin{aligned}
\sigma_{11} &= R^{-1}\mathrm{Re}\{-\alpha\beta_2\chi_1(z_1) + \beta\beta_2\chi_1(z_2) + \alpha\beta\chi_2(z_1) - \beta_1\beta_2\chi_2(z_2)\},\\
\sigma_{12} &= R^{-1}\mathrm{Im}\{\beta_1\beta_2\chi_1(z_1) - \beta^2\chi_1(z_2) - \beta\beta_1[\chi_2(z_1) - \chi_2(z_2)]\},\\
\sigma_{22} &= R^{-1}\mathrm{Re}\{\beta\beta_2[\chi_1(z_1) - \chi_1(z_2)] - \beta^2\chi_2(z_1) + \beta_1\beta_2\chi_2(z_2)\},\\
u_1 &= \frac{c}{2\mu R}\,\mathrm{Re}\{\beta_2\chi_1(z_1) - \beta\beta_2\chi_1(z_2) - \beta\chi_2(z_1) + \beta_1\beta_2\chi_2(z_2)\},\\
u_2 &= \frac{c}{2\mu R}\,\mathrm{Im}\{-\beta_1\beta_2\chi_1(z_1) + \beta\chi_1(z_2) + \beta\beta_1\chi_2(z_1) - \beta_1\chi_2(z_2)\},
\end{aligned}
\qquad (1.5)
$$

where $R = R(c) = \beta_1\beta_2 - \beta^2$ is a function which is proportional to the Rayleigh function (c_R is a unique positive root of $R(c) = 0$), $\beta_j = \sqrt{1 - c^2/c_j^2}$, $z_j = x = i\beta_j y$ ($j = 1, 2$), c_1 and c_2 are the velocities of expansion and shear waves, respectively, μ is the shear modulus, $\beta = 1/2(1 + \beta_2^2)$, and $\alpha = 1 + \beta_1^2 - \beta$.

We shall give for reference the equations which relate the second derivatives of ordinary complex expansion and shear waves (complex potentials of the displacements) $\Phi_j(z_j)$ with the functions χ_j:

$$2\mu R\Phi_1''(z_1) = -\beta_2\chi_1(z_1) + \beta\chi_2(z_1), \quad 2\mu R\Phi_2''(z_2) = -i\beta\chi_1(z_2) + i\beta_1\chi_2(z_1).$$

On the cut $z_1 = z_2 = x$ the expressions

$$\sigma_{12}(x, 0) = \mathrm{Im}\{\chi_1(x)\}, \ \sigma_{22}(x, 0) = \mathrm{Re}\{\chi_2(x)\} \qquad (1.6)$$

follow from (1.5).

The substitution of (1.6) into (1.1) with (1.3) and (1.5) taken into account lead to the Hilbert boundary-value problem: to find two analytic functions $\chi_1(z)$ and $\chi_2(z)$ of the complex variable in the upper half-plane which satisfy the conditions

$$
\begin{gathered}
\mathrm{Im}\chi_1 = 0 \ (|x| < \infty), \ \mathrm{Im}\chi_2 = 0 \ (x \in L_1),\\
\mathrm{Re}\,\chi_2 = \sigma_\infty \ (x \in L - L_0), \ \mathrm{Re}\,\chi_2 = \sigma_\infty - \sigma(x) \ (x \in L_0),
\end{gathered}
\qquad (1.7)
$$

on the real axis and the limits $(j = 1, 2)$

$$|\chi_j(z)| < \frac{\text{const}}{|z \mp d|^{1/2}} \ (z \to \pm d + 0i), \ |\chi_j(z)| < \frac{\text{const}}{|z|} \ (|z| \to \infty) \tag{1.8}$$

at the singular points and at infinity.

In addition the functions $\chi_j(z)$ should be such that the stresses and velocities calculated in terms of them by means of (1.5) and (1.6) do not contradict the inequalities (1.2).

2. We shall construct a unique solution of the Hilbert problem with the discontinuous coefficients (1.7) and (1.8) (we shall assume the parameters d and δ_1 to be unknown for the present) by the method of [2]. In the sub-Rayleigh regime of motion of the load $(c < c_R)$ the requirements (1.2) and (1.4) dictate the absence of singularities at the points $z = \pm d$. In the opposite case unbounded tensile stresses would appear to the right of the slit as in crack theory (if one assumes a singularity at the point $z = d$ and satisfies the requirement $f(x) \geqslant 0$) or the energy flow w_1 would be negative (a singularity at the point $z = -d$).

One can make use for the determination of the sign of w_k of the expression of this quantity in terms of the coefficients of the stress intensity, the velocity c (with the direction of opening of the cut taken into account), and the constant parameters of the medium, which are given, for example, in [6], from which it follows that $\operatorname{sgn} w_k = (-1)^k \operatorname{sgn} R$ $(k = 1, 2)$.

One can write the general solution of the problem (1.7) and (1.8) in the form

$$\chi_1(z) = 0, \quad \chi_2(z) = \frac{i}{\pi} \int\limits_{\delta_2}^{\delta_1} \left(\frac{d^2 - z^2}{d^2 - t^2}\right)^{1/2} \frac{\sigma(t)\,dt}{t - z} - \frac{i\sigma_\infty}{\pi} \int\limits_{-d}^{d} \left(\frac{d^2 - z^2}{d^2 - t^2}\right)^{1/2} \frac{dt}{t - z} + \frac{B}{(d^2 - z^2)^{1/2}},$$

and due to what has been said above we set the constant B equal to zero. The cut joins the branch points $z = \pm d$ on the real axis to make the radical uniform; the radical takes positive values on the upper edge of the cut.

The second integral in the expression for $\chi_2(z)$ reduces to the integral [7]

$$\frac{1}{\pi} \int\limits_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}\,(t - z)} = -\frac{1}{\sqrt{z^2 - 1}}_{z = x \in L_1} = -\frac{\operatorname{sgn} x}{\sqrt{x^2 - 1}}, \tag{2.1}$$

therefore the solution takes the simpler form

$$\chi_1(z) = 0, \quad \chi_2(z) = \frac{i}{\pi} \int\limits_{\delta_2}^{\delta_1} \left(\frac{d^2 - z^2}{d^2 - t^2}\right)^{1/2} \frac{\sigma(t)\,dt}{t - z} + \sigma_\infty \quad (z \in D). \tag{2.2}$$

The narrowing of (2.2) to the real axis with the application of the Sokhots–Plemel' formulas [2] leads to the expressions

$$\begin{gathered} \chi_1(x) = 0, \ \chi_2(x) = \\ = \begin{cases} \sigma_\infty - \sigma(x) + ig(x) & (x \in L, \ \sigma(x) \equiv 0 \ \text{for} \ x \in L - L_0), \\ \sigma_\infty + g(x) \operatorname{sgn} x & (x \in L_1), \end{cases} \\ g(x) = \frac{1}{\pi} \int\limits_{\delta_1}^{\delta_2} \left|\frac{d^2 - x^2}{d^2 - t^2}\right|^{1/2} \frac{\sigma(t)\,dt}{t - x} \quad (x \in L + L_1). \end{gathered} \tag{2.3}$$

Here and below the integrals are understood in the sense of the principal value if they do not exist in the Riemannian sense.

The substitution of (2.2) and (2.3) into formulas (1.5) will give the solution of the posed problem in D and on $L_1 + L$ only after the unknown quantities d and δ_1 are determined (the coordinate δ_2 will then become known, since the size of the loading zone, which is equal to $\delta_2 - \delta_1$, is naturally specified). In addition one should verify the conditions (1.2). With these goals as well as with the goal of studying the singularities of the solution we shall write out the formulas for the total stress $\sigma^0_{22}(x, 0)$ outside the slit, the velocities $u_2(x, 0)$, and the opening quantities $f(x)$ obtained with the use of (1.1), (1.5), and (2.3) and represent the asymptotic behavior of these functions in one-sided neighborhoods of the singular point $x = d$ (similar behavior will occur when $x \to -d$):

$$\sigma_{22}^0 = g(x)\,\mathrm{sgn}\,x < 0 \quad (x \in L_1), \quad \sigma_{22}^{0'} = -C\sqrt{x-d} + O\left((x-d)^{3/2}\right) \quad (x \to d+0),$$
$$u_2 = -cbg(x) = -c(df/dx) \ (x \in L), \ u_2 = cbC\sqrt{d-x} + O((d-x)^{3/2}) \quad (x \to d-0), \tag{2.4}$$
$$f(x) = b\int_x^d g(t)\,dt, \quad f(x) \sim \frac{2}{3}cbC(d-x)^{3/2} \quad (x \to d-0),$$
$$C = \frac{\sqrt{2d}}{\pi}\int_{\delta_2}^{\delta_1}\frac{\sigma(t)\,dt}{(d-t)^{3/2}(d+t)^{1/2}} > 0, \quad b = \frac{\beta_1(1-\beta)}{2\mu R} \quad (b > 0 \text{ for } 0 < c < c_R).$$

Formulas (2.4) show that the stresses and velocities are continuous at the points $x_{1,2}$. A boundary condition is used in (2.4) which follows from the continuity of the shifts $f(+d) = 0$; another similar boundary condition $f(-d) = 0$ generates a single equation for finding the constants d and δ_1:

$$\int_{-d}^{d} g(t)\,dt = 0. \tag{2.5}$$

We obtain another equation by satisfying the condition of disappearance at infinity. It follows from (2.2) that

$$\chi_2(z) = \sigma_\infty - C_1 + O(1/z) \qquad (|z| \to \infty) \tag{2.6}$$

and it must be that

$$C_1 \equiv \frac{1}{\pi}\int_{\delta_1}^{\delta_2}\frac{\sigma(x)\,dx}{(d^2-x^2)^{1/2}} = \sigma_\infty. \tag{2.7}$$

The solutions of the nonlinear Eqs. (2.5) and (2.7) should satisfy the adopted conditions $\delta_1, \delta_2 \in L$, which imposes constraints on the permissible functions $\sigma(x)$.

Below for illustration we shall consider a particular form of a load which has a point of symmetry on the x axis. Then it is natural to seek a solution which is symmetric with respect to the y axis ($\delta_1 = -\delta_2$). Equation (2.5) is automatically satisfied, since $g(x)$ becomes an antisymmetric function, and Eq. (2.6) has a unique root $d > \delta_2$ upon the satisfaction of the following constraint on $\sigma(x)$ and δ_2:

$$\frac{1}{\pi}\int_{-\delta_2}^{\delta_2}\frac{\sigma(x)\,dx}{(\delta_2^2-x^2)^{1/2}} > \sigma_\infty,$$

which is simultaneously a condition for the applicability of the scheme under discussion in the case of symmetric loads.

We note that for such loads one should replace $O(1/z)$ in (2.6) by $O(1/z^2)$. This is related to the fact that the moment of the forces applied to the elastic half-plane vanishes [2].

For a rectangular load ($\sigma(x)) = \sigma_0$ const $> \sigma_\infty$, $|x| < \delta_2$) we obtain from (2.7)

$$d = \delta_2/\sin(\pi\sigma_\infty/2\sigma_0). \tag{2.8}$$

Setting $\sigma_0 = P_0/(2\delta_2)$ in (2.8) and letting δ_2 tend to zero, we obtain the result for the slit half-length in the case of a concentrated force $d = P_0/(\pi\sigma_\infty)$.

As $\delta_2 \to d$, $\sigma_0 \to \sigma_\infty$ follows from (2.8), and then $g(x) \to 0$, $f(x) \to 0$, and $u_2(x) \to 0$ follow from (2.3) and (2.4). One can apply this result to such a physical situation. If as $t \to \infty$ one considers the limit of expansion of a definite portion of a barotropic gas from the point $x = (0, 0)$ (an explosion) on a cut in an elastic half-plane, degeneracy of the parameters of the expanding slit and the pressure will evidently be as follows:

$$d(t) \to \infty, \quad \sigma(x, t) \to \sigma_\infty, \quad f(x, t) \to 0, \quad \int\limits_{-d(t)}^{d(t)} f(x, t)\, dx \to \text{const.}$$

Now we shall turn our attention to the fact that the parameters d and δ_1 as the solutions of (2.5) and (2.7) do not depend on the velocity c (the fact that the stresses σ_{12} and σ_{22} on the cut do not depend on c is known [5]). Therefore the results referring to a cut which have been obtained are simply transferred to statics. Only the shape of the slit $f(x)$ depends through the coefficient b on c ($b(c) \to \infty$ and $c \to c_R$ are the manifestation of the resonance properties of the elastic body).

For the transition to statics we find the limit

$$\lim_{c\to 0} \frac{z_2 - z_1}{R} = \lim_{c\to 0} iy \frac{\beta_2 - \beta_1}{R} = -iy \tag{2.9}$$

inside the region and represent $\chi_2(z_2)$ in the form of a series

$$\chi_2(z_2) = \chi_2(z_1) + (z_2 - z_1)\chi_2'(z_1) + \ldots$$

Let us substitute the last expansion into (1.5) with account taken of $\chi_1(z_j) \equiv 0$, $j = 1, 2$, and proceed to the limit $c \to 0$, bearing (2.9) in mind. We shall obtain formulas for the stresses in the static version of the problem

$$\sigma_{11} = \operatorname{Re}\{\chi_2(z) + iy\chi_2'(z)\}, \quad \sigma_{12} = -\operatorname{Re}\{y\chi_2'(z)\},$$
$$\sigma_{22} = \operatorname{Re}\{\chi_2(z) - iy\chi_2'(z)\}.$$

We note that $y\chi_2'(z) \to 0$ as $y \to 0$, $x = \text{const}$ as $|z| \to \infty$, and the function $\chi_2(z)$ is determined as previously by the equality in (2.2).

In the range of velocities c under discussion the load does not produce work (which is rather evident for the symmetric case); there is no energy drain: there is no radiation to infinity (for this the order of the stresses as $|x| \to \infty$ should be equal to $O(|x|^{-1/2})$), and the fluxes w_k are equal to zero.

Remazk. One can arbitrarily formulate the problem in the velocity range $0 < c < c_R$ with simpler but mor rigorous constraints, requiring, for example, continuity of the stresses at the ends of the slit, as in problems concerning punches without angular points [2, 5]. However, the question arises of the uniqueness of the solution in a broader but all the same physically acceptable class of functions (the conditions (1.4) instead of the continuity conditions). A positive answer to this question is given here. It preserves its validity and is applicable to the problems of punches mentioned above if one puts in them in addition a condition that the surface of the elastic medium not intersect with the punch surface outside the contact zone.

For comparison we shall consider the problem of the motion of a crack under conditions (1.1)-(1.4). Now one should only give up the first condition (1.2) and allow tensile stresses for $x > d$, $y = 0$. Let us write out the final form of the solution ($\chi_1(z) \equiv 0$):

$$\chi_2(z) = \frac{i}{\pi} \int\limits_{\delta_1}^{\delta_2} \frac{A_0(z)}{A_0(t)} \frac{\sigma(t)\, dt}{t - z} + [1 + iA_0(z)]\, \sigma_\infty, \tag{2.10}$$

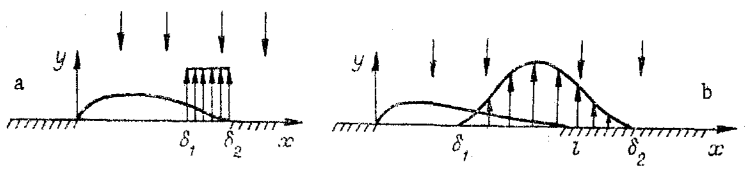

Fig. 2

$$\chi_2(x)=\begin{cases}\sigma_\infty-\sigma(x)+ig(x), & x\in L,\\ -g(x)+\sigma_\infty, & x\in L_1,\end{cases}\quad g(x)=\frac{1}{\pi}\int_{\delta_1}^{\delta_2}\left|\frac{A_0(x)}{A_0(t)}\right|\frac{\sigma(t)\,dt}{t-x}+|A_0(x)|\,\sigma_\infty,$$

where the radical $A_0(z) = (d + z)^{1/2}(d - z)^{-1/2}$ with the previous uniformity rule is selected as the uniform canonical solution of the Hilbert problem (1.7) and (1.8) [2].

The behavior of the solution in a small neighborhood of the point $\mathbf{x_1}$ is analogous to the case of a slit, and the stress $\sigma^0_{22}(x, 0)$ for $x > 0$ takes the usual asymptotic form for crack theory

$$\sigma^0_{22}\sim N\left(\frac{x}{d}-1\right)^{-1/2},\quad x\to d+0,\ N=\frac{\sqrt{2}}{\pi}\int_{\delta_1}^{\delta_2}\frac{\sigma(t)\,dt}{(d^2-t^2)^{1/2}}-\sqrt{2}\sigma_\infty.$$

$N\geqslant 0$ follows from the nonintersection condition (1.2). When $N > 0$, the parameter δ_1 is either simply specified or one can attempt to determine it from the breakdown criterion for fixed c and $\sigma(x)$ (we shall not specify them here). In any case Eq. (2.6) determines the half-length of the crack d, which is smaller than the corresponding half-length of the slit. The load when $N\neq 0$ produces power, which is liberated at the tip of the crack.

Setting $N = 0$, we again arrive at Eq. (2.7) and the solution (2.2). Transformation of (2.10) into (2.2) under the condition $N = 0$ is completed similarly [2].

3. We shall construct the solution of the problem in the super-Rayleigh region of rates of movement of the load ($c_R < c < c_2$). Preliminary analysis of the possible situations based on taking account of the conditions (1.2)-(1.4) shows that one should now arrange the load in a significantly asymmetric way (Figs. 2a, b). Let us place the origin of coordinates at the end of the slit and denote the full length of the slit by the letter "l," and we obtain

$$L=\{0<x<l\},\ L_0=\{\delta_1<x<\delta_2\},\ L_1=\{x\notin L,\ x\neq 0,\ l\}.$$

We select $A(z) = [(l-z)/z]^{1/2}$ with the previous uniformity rule of the radical as the canonical uniform solution used in construction of the Hilbert problem (1.7) [2].

A unique solution of the problem (1.7) which does not contradict the conditions (1.2) and (1.4) and limits similar to (1.8) is written in the form

$$\chi_1(z)=0,\ \chi_2(z)=\frac{i}{\pi}\int_{\delta_1}^{l}\frac{A(z)}{A(t)}\frac{\sigma(t)\,dt}{t-z}+[1-iA(z)]\,\sigma_\infty\qquad(z\in D),$$

$$\chi_2(x+i0)=\begin{cases}\sigma_\infty-\sigma(x)+ig(x) & (x\in L),\ \sigma(x)\equiv 0\qquad (x\notin L_0),\\ g(x)+\sigma_\infty & (x\in L_1),\end{cases}\tag{3.1}$$

$$g(x)=\frac{a(x)}{\pi}\int_{\delta_1}^{l}\frac{\sigma(t)\,dt}{a(t)(t-x)}-B(x),\ a(x)=\left|\frac{l-x}{x}\right|^{1/2},$$

$$B(x)=\sigma_\infty a(x)\qquad(x\in L+L_1).$$

as a result of the calculation of a single integral analogous to (2.1).

The expressions $u_2(x)$ and $f(x)$ in terms of $g(x)$ will remain as before. Only the sign of $b(c)$ is changed, since when passing through the Rayleigh velocity $R(c)$ changes its own sign to negative.

First we shall consider a load with a discontinuity on the leading edge. Then one should superpose the coordinate of the edge with the coordinate of the tip of the slit $x = l$ from a consideration of the need to satisfy the conditions (1.1) and (1.2), and only a single undetermined parameter l remains in the problem. There is an equation similar to (2.5) for fixing the length of the opening:

$$\int_0^l g(x)\,dx = 0. \tag{3.2}$$

We shall calculate the integrals in the solution of the problem (3.1) for a stepwise load $\sigma(x) = \sigma_0 = \text{const}$, $x \in L_0$. The integral in the expression for g(x) is calculated with the help of substitution. We shall give the final result, temporarily taking the size of the loading zone as the unit of length and the amplitude of the dynamic load as the unit of stress ($\sigma_\infty < 1$, $\delta_1 = l - 1$):

$$g(x) = ka(x) + \begin{cases} \frac{1}{\pi}\ln\left|\frac{a(x)+a(l-1)}{a(x)-a(l-1)}\right|, & x \in L, \\ -1 + \frac{2}{\pi}\operatorname{arctg}[a(x)/a(l-1)], & x \in L_1, \end{cases} \tag{3.3}$$
$$k = 1 - \frac{2}{\pi}\operatorname{arctg}(l-1)^{1/2} - \sigma_\infty.$$

We have a bit to say in connection with the existence of a root of Eq. (3.2) for the function g(x) specified by formula (3.3). It is easy to establish that as $l \to 1$ and $l \to \infty$ the function g(x) is consequently positively and negatively determined almost everywhere on L. Therefore in the interval $1 < l < \infty$ a unique root should evidently exist, which is confirmed by calculation. A plot of $l = l(\sigma_\infty)$ on the basis of (3.2) and (3.3) is given in Fig. 3. We note one inequality which it is necessary for the root of (3.2) to satisfy in the case of (3.3):

$$k(l, \sigma_\infty) < 0 \Rightarrow l > 1 + \operatorname{ctg}[(\pi/2)\sigma_\infty] \to \infty, \ \sigma_\infty \to 0. \tag{3.4}$$

Analytically continuing g(x) from (3.3) with $x \in L_1$ into the region D, we shall establish the final form of the solution for a load of rectangular shape:

$$\chi_2(z) = ikA(z) + \sigma_\infty - 1 + \frac{2}{\pi}\operatorname{arctg}\left[\frac{iA(z)}{a(l-1)}\right]_{|z|\to\infty} = O\left(\frac{1}{z}\right).$$

Let us clarify the behavior of the main functions on the cut and verify the satisfaction of the conditions (1.2). We obtain the asymptotic consequences of (3.3) in advance:

$$g(x)_{x\to\pm 0} = k|l/x|^{1/2} + O(|x/l|^{1/2})_{x\to l-0} = M\sqrt{1-(x/l)} + O(((x/l)-1)^{3/2}), \tag{3.5}$$
$$g(x) = -1 + M\sqrt{(x/l)-1} + O(((x/l)-1)^{3/2}) \ (x \to l+0),$$
$$M = k + \frac{2}{\pi}(l-1)^{1/2} > 0.$$

Proceeding from (1.5), (1.6), (3.3), and (3.5), we have

$$\sigma_{22}^0 = g(x) < 0 \quad (x \in L_1), \tag{3.6}$$
$$f(x) = b\int_0^x g(t)\,dt_{x\to+0} \sim 2bk(x/l)^{1/2}_{x\to l-0} \sim -\frac{2}{3}lbM\left(1-\frac{x}{l}\right)^{3/2}$$

at y = 0.

One can establish by taking (3.2) into account that the condition $f(x) \geqslant 0$ is satisfied on L. Thus all the previously made assumptions are justified.

As follows from (3.5) and (3.6), the stress σ_{22} (and σ_{11}) is not bounded near and to the left of the point x = 0. One can compare the stress state here with the state near the tip

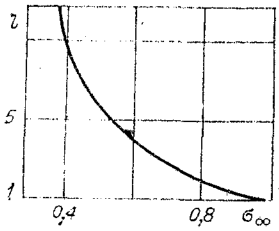

Fig. 3

of the crack upon its sub-Rayleigh motion or in statics, and it is similar to the stress state in the neighborhood of the contact point upon the collision of elastic strips in the super-Rayleigh subsonic region [4]. The leading term of the asymptote as $x \to l + 0$ for the function $\sigma_{22}^{0}(x)$ is a constant which is produced by the nonzero boundary conditions at the tip of the slit and which is the contribution of the nonuniform solution to the asymptotic expansion of this function in the neighborhood of the singular point. It is interesting to note that in this case this locally determined asymptote dominates over the contribution made by uniform and nonlocally determined solutions in the neighborhood of this same point (the second term in the expansion of $\sigma_{22}^{0}(x$, as $x \to l + 0$ in (3.5) and (3.6)). Concerning the kinematic characteristics of the slit as $x \to l - 0$, they are the same as in the sub-Rayleigh regime.

A local analysis shows that if one shifts the load (with a discontinuity) to the left or to the right of the point $x = l$, one of the conditions (1.1) and (1.2) is not satisfied. This proves the uniqueness of the choice of the scheme for positioning this load relative to the slit.

Now let $\sigma(x)$ be a function which is doubly continuously differentiable nonnegative on L_0 and which vanishes at the ends of L_0. Then it becomes necessary to determine two numerical parameters: l and δ_1. We shall perform further analysis, assuming for the sake of definiteness that the load $\sigma(x)$ results from an external field (it is natural to assume that an explosive load was formed from within, for example, from a plane explosion, and the smooth load was formed from fields incident on the cut from outside; this affects the calculation of $\sigma_{22}^{0}(x, y)$ when checking the condition of no separation). It is more correct in this case to assume that $\sigma(x)$ extends to infinitely in both directions without allowing the existence of weak discontinuities propagating with wave velocities. But for the sake of simplicity we shall assume that the specified $\sigma(x)$ is a sufficiently good approximation of the actual loading.

Isolating the singularity of the integrand in the expression for $g(x)$ from (3.2), we bring it to the form

$$
\begin{gathered}
g(x) = \frac{a(x)}{\pi} \int_{\delta_1}^{l} \frac{\sigma(t) - \sigma(x)}{a(t)(t - x)}\, dt + \sigma_*(x)\, a(x) + \frac{\sigma(x)}{\pi} \ln \left| \frac{a(x) + a(\delta_1)}{a(x) - a(\delta_1)} \right| \quad (x \in L), \\
g(x) = M(x; l, \delta_1)\, a(x) - \sigma(x) \quad (x \in L_1), \quad \sigma(x) \equiv 0 \quad (x \notin L_0), \\
M = \frac{1}{\pi} \int_{\delta_1}^{l} \frac{\sigma(t) - \sigma(x)}{a(t)(t - x)}\, dt + \sigma_*(x) + \frac{2\sigma(x)}{\pi a(x)} \operatorname{arctg} \frac{a(x)}{a(\delta_1)}, \\
\sigma_*(x) = \sigma(x) \left[1 - \frac{2}{\pi} \operatorname{arctg} a^{-1}(\delta_1) \right] - \sigma_\infty.
\end{gathered}
\tag{3.7}
$$

We shall write down a formula for $\sigma_{22}^{0}(x, 0)$ with account taken of the fact that now the total stress field is the superposition of three components: a uniform static field, an external variable field (which would exist in the absence of the cut), and one perturbed by the presence of the slit:

$$
\sigma_{22}^{0} = -\sigma_\infty + \sigma(x) + \sigma_{22}(x, 0) = a(x) \cdot M(x; l, \delta_1) \qquad (x \in L_1).
$$

The inequality

$$
M(x; l, \delta_1) \leqslant 0 \tag{3.8}
$$

should be satisfied, and in particular,

$$M(l;\ l,\ \delta_1)\leqslant 0, \tag{3.9}$$

in order that the closing requirement be satisfied locally.

At the same time if one calculates the asymptote of g(x) to the left of the point $x = l$, the requirement that the nonintersection condition of the edges

$$df/dx = bg(x) = b[M(l;\ l,\ \delta_1)a(x) + O(a^2(x))] \leqslant 0 \qquad (b < 0)$$

be locally satisfied leads to the inequality

$$M(l;\ l,\ \delta_1)\geqslant 0. \tag{3.10}$$

The solution of the system of inequalities (3.9) and (3.10) is the equality

$$M(l;l,\delta_1)\equiv\frac{1}{\pi}\int\limits_{\delta_1}^{l}\frac{[\sigma(l)-\sigma(t)]t^{1/2}}{(l-t)^{3/2}}dt+\sigma_*(l)+\frac{2}{\pi}\frac{\sigma(l)}{a(\delta_1)}=0, \tag{3.11}$$

which closes the system of nonlinear Eqs. (3.2) and (3.11) for finding the parameters l and δ_1. A qualitative investigation of this system under the condition $\max_{x\in L_0}\sigma(x) > \sigma_\infty$ and uniqueness of the extremum of $\sigma(x)$ on L_0 shows that it should have a solution and with a value of l lying to the right of the point at which $\sigma(x)$ has a maximum (see Fig. 2b).

One should note that (3.11) is a condition for the inversion of the leading term of the asymptote of the stress and velocity, field in a small neighborhood of the singular point $x = l$, $y = 0$. Thus the solution has a weaker singularity at this point than in the versions of the problem discussed earlier. The behavior of the functions on the cut near $x = l$ is of the form

$$\begin{gathered}\sigma_{22}^0(x)\sim \mathrm{const}\,(x-l)^{3/2} \qquad (x\to l+0),\\ df/dx\sim -b\,\mathrm{const}\,(l-x)^{3/2},\ f(x)\sim b\,\mathrm{const}\,(l-x)^{5/2}\ (x\to l-0).\end{gathered} \tag{3.12}$$

In the derivation of (3.12) for an estimate of the difference in the integrals which arises in evaluating the expression $M(x;\ l,\ \delta_1) - M(l;\ l,\ \delta_1)$, the differentiability of the integrand over x for $x \geqslant l$ in (3.7) has been proven, and a Taylor's formula with a residual term in the Peano form [8] has been used, along with the smoothness condition of $\sigma(x)$ formulated above.

The asymptotic behavior in the neighborhood of the other edge of the slit will be the same as in the case of a discontinuous load. A positive flow of energy to the point $x = 0$ corresponds to it [6]:

$$w_1=-\frac{c^3\beta_1K_I^2}{8\mu c_2^2R},\quad K_I=\lim_{x\to -0}\left[\sigma_{22}(x,0)\sqrt{2\pi x}\right],\quad R<0.$$

It is equal to the power developed by the load (due to the asymmetry of its application this power is now different from zero). Such is the energy balance in the system for $c_R < c < c_2$.

One can establish that $\sigma_{22}^0(x,\ 0) < 0$ when $x < 0$. If the root $(l,\ \delta_1)$ of the system (3.2) and (3.11) is unique or, for safety's sake the smallest positive l among the roots of (3.2) and (3.11) is selected, the nonintersection condition is satisfied. The situation is worse with the checking of the closing condition to the right of the slit (3.8). One can check it asymptotically:

$$M(x) = -\sigma_\infty + O(1/x),\ x\to\infty,$$

and for functions $\sigma(x)$ which are convex upwards ($\sigma''(x) \leqslant 0$, $x \in L_0$) one can show that $M'_x|_{x=l} < 0$. This inequality along with (3.12) shows that immediately beyond the tip of the slit on its continuation the stresses $\sigma_{22}^0(x,\ 0)$ are compressive. Concrete calculations easily accomplished with the help of a computer are necessary for global verification of this condition. One should naturally alter the separation scheme if this condition is not satisfied.

It is evident from the discussion that wave effects are noticeably expressed at near-Rayleigh rates of movement of the load (just as in other similar problems incidentally). The nature of the singularities of the solution at the tips of the slit changed upon the transition through the velocity of a Rayleigh wave; for smooth loads at one of them the singularity was replaced by a stronger one and at the other — by a weaker one.

The problem of the separation of an elastic medium pressed to a rigid plane base is solved simultaneously.

The solution of the time-independent problem is justified if it is a limit of the corresponding time-dependent problem, and this will be so if the velocity of the loads does not coincide either with one of the velocities of intrinsic resonance waves of the elastic system (waves for which the phase velocity is equal to the group velocity) [9]. Such cases are excluded in advance in this discussion ($c \neq c_R$, c_1, c_2).

The question of the separation of an elastic strip from the base under the action of a force moving at a constant sub-Rayleigh velocity has been discussed in [10] (also see the bibliography for it).

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